0 Introduction

Four-dimensional webs $W(3,2,2)$ have been considered in many books and papers (see, for example, the books [AS 92], [G 88] and the papers [B 35], [C 36], [G 85, 86, 87, 99], [K 81, 83, 84, 96]). They are of special interest since

a) They are the first webs generalizing the notion of two-dimensional three-web introduced by Blaschke [Bl 28] to higher codimension (see [B 35]).

b) They provide examples illustrating different properties of webs (see [AS 92], [B 35], [C 36], [G 88], [G 99]).

c) Their torsion tensor has a simple structure: $a^i_{jk} = a_{[j}\delta^i_{k]}$, where a_i is a covector (see [AS 92], [G 88]).

d) They are connected with the pseudoconformal structures $CO(2,2)$ of signature $(2,2)$ (see [AG 96], [AG 99], [K 81, 83, 96]).

If the covector $a = \{a_1, a_2\}$ of a web $W(3,2,2)$ does not vanish, then it defines a transversal a-distribution invariantly and intrinsically connected with a web. In general, this a-distribution is not integrable.

In Section 1 we find necessary and sufficient conditions of its integrability and prove the existence theorem for webs $W(3,2,2)$ with integrable transversal a-distributions (see Theorems 1 and 3).

In Section 2 we prove that for a web $W(3,2,2)$ with the integrable distribution Δ, its integral surfaces V^2 are geodesicly parallel in an affine connection of a certain bundle of affine connections (Theorem 4 (i)) and study three-webs for which the surfaces V^2 are geodesicly parallel with respect to affine connections of this bundle (Theorem 4 (ii)).

In Section 3, we find conditions for webs $W(3,2,1)$ cut by the foliations of $W(3,2,2)$ on V^2 to be hexagonal (Theorem 6) and prove the existence theorem for such webs $W(3,2,2)$ (Theorem 7). We also prove the existence theorem for webs $W(3,2,2)$ of the subclass which is the intersection of subclasses considered in Sections **2** and **3** (Theorem 8), and establish some properties of webs $W(3,2,2)$ implied by a relationship existing between four-dimensional three-webs and pseudoconformal structures $CO(2,2)$ of signature $(2,2)$ (Theorem 9).

In addition, in Sections **2** and **3** we find an analytic characterization of three-webs considered in these sections not only in a specialized frame but also in the general frame.

Note that webs $W(3,2,2)$ with integrable transversal a-distributions as well as webs $W(3,2,2)$, for which integral surfaces V^2 of Δ are geodesicly parallel in an affine connection of a certain bundle of affine connections, and webs $W(3,2,2)$, for which the three-subwebs $W(3,2,1)$ cut by the foliations of $W(3,2,2)$ on V^2 are hexagonal, are considered in this paper for the first time.

1 The transversal distribution of a web $W(3,2,2)$

1. The leaves of the foliation $\lambda_u,\ u = 1,2,3$, of a web $W(3,2,2)$ are determined by the equations $\underset{u}{\omega}^i = 0,\ i = 1,2$, where

$$\underset{1}{\omega}^i + \underset{2}{\omega}^i + \underset{3}{\omega}^i = 0 \tag{1}$$

(see, for example, [G 88], Section **8.1** or [AS 92], Section **1.3**). The forms $\underset{1}{\omega}^i$ and $\underset{2}{\omega}^i$ are basis forms on a manifold M^4 carrying the web $W(3,2,2)$.

The structure equations of such a web can be written in the form

$$\left\{ \begin{array}{l} d\underset{1}{\omega}^i = \underset{1}{\omega}^j \wedge \omega_j^i + a_j \underset{1}{\omega}^j \wedge \underset{1}{\omega}^i, \\ d\underset{2}{\omega}^i = \underset{2}{\omega}^j \wedge \omega_j^i - a_j \underset{2}{\omega}^j \wedge \underset{2}{\omega}^i. \end{array} \right. \tag{2}$$

The differential prolongations of equations (2) are (see [G 88], Sections **8.1** and **8.4** or [AS 92], Section **3.2**):

$$d\omega_j^i - \omega_j^k \wedge \omega_k^i = b_{jkl}^i \underset{1}{\omega}^k \wedge \underset{2}{\omega}^l, \tag{3}$$

$$da_i - a_j \omega_i^j = p_{ij} \underset{1}{\omega}^j + q_{ij} \underset{2}{\omega}^j, \tag{4}$$

where

$$b_{[j|l|k]}^i = \delta_{[k}^i p_{j]l}, \quad b_{[jk]l}^i = \delta_{[k}^i q_{j]l}. \tag{5}$$

The quantities

$$a_{jk}^i = a_{[j} \delta_{k]}^i \tag{6}$$

and b_{jkl}^i are the *torsion and curvature tensors* of a three-web $W(3,2,2)$. Note that for webs $W(3,2,2)$ the torsion tensor a_{jk}^i always has structure (6), where $a = \{a_1, a_2\}$ is its transversal covector. If $a = 0$, then a web $W(3,2,2)$ is isoclinicly geodesic. Such webs were studied in [A 69]. In what follows, *we will assume that $a \neq 0$, i.e., a web $W(3,2,2)$ is nonisoclinicly geodesic.*

The covector a_i is defined in a second-order differential neighborhood of a point $x \in M^4$, and the curvature tensor b_{jkl}^i as well as the tensors p_{ij} and q_{ij} are defined in a third-order neighborhood of $x \in M^4$. By conditions (5), the tensor b_{jkl}^i can be represented in the form

$$b_{jkl}^i = s_{jkl}^i + \frac{2}{3} p_{jk} \delta_l^i - \frac{1}{3} p_{kl} \delta_j^i - \frac{1}{3} p_{lj} \delta_k^i - \frac{1}{3} q_{jk} \delta_l^i - \frac{1}{3} q_{kl} \delta_j^i + \frac{2}{3} q_{lj} \delta_k^i,$$

where $s_{jkl}^i = b_{(jkl)}^i$ is the symmetric part of the tensor b_{jkl}^i (see [AS 92], p. 113). The last formula implies that in a third-order neighborhood of $x \in M^4$, there are 8 independent components of the tensors p_{ij} and q_{ij} and also 8 independent components of the tensor b_{jkl}^i.

In this paper we will need the differential prolongations of equations (3), (4), and (5). They have the form

$$[\nabla b^i_{jkl} + b^i_{jkl} a_m (\underset{1}{\omega}^m - \underset{2}{\omega}^m)] \wedge \underset{1}{\omega}^k \wedge \underset{2}{\omega}^l = 0, \tag{7}$$

$$(\nabla p_{jk} + p_{jk} a_l \underset{1}{\omega}^l) \wedge \underset{1}{\omega}^k + (\nabla q_{jk} - q_{jk} a_l \underset{2}{\omega}^l) \wedge \underset{2}{\omega}^k + a_m b^m_{jkl} \underset{1}{\omega}^k \wedge \underset{2}{\omega}^l = 0, \tag{8}$$

where

$$\begin{aligned} \nabla b^i_{jkl} &= db^i_{jkl} - b^i_{mkl}\omega^m_j - b^i_{jml}\omega^m_k - b^i_{jkm}\omega^m_l + b^m_{jkl}\omega^i_m, \\ \nabla p_{jk} &= dp_{jk} - p_{mk}\omega^m_j - p_{jm}\omega^m_k, \\ \nabla q_{jk} &= dq_{jk} - q_{mk}\omega^m_j - q_{jm}\omega^m_k. \end{aligned}$$

Equations (7) and (8) prove that the forms $\nabla b^i_{jkl}, \nabla p_{jk}$, and ∇q_{jk} are linear combinations of the basis forms $\underset{1}{\omega}^k$ and $\underset{2}{\omega}^k$:

$$\left\{ \begin{array}{l} \nabla b^i_{jkl} = \overline{b}^i_{jklm} \underset{1}{\omega}^m + \widetilde{b}^i_{jklm} \underset{2}{\omega}^m, \\ \nabla p_{jk} = \overline{p}_{jkl} \underset{1}{\omega}^l + \widetilde{p}_{jkl} \underset{2}{\omega}^l, \\ \nabla q_{jk} = \overline{q}_{jkl} \underset{1}{\omega}^l + \widetilde{q}_{jkl} \underset{2}{\omega}^l. \end{array} \right. \tag{9}$$

Substituting decompositions (9) into equations (7) and (8) and using the linear independence of the forms $\underset{\alpha}{\omega}^i$, we find that the coefficients in (9) satisfy the following conditions:

$$\left\{ \begin{array}{l} \overline{b}^i_{j[k|l|m]} + a_{[m} b^i_{|j|k]l} = 0, \\ \widetilde{b}^i_{jk[lm]} - a_{[m} b^i_{|jk|l]} = 0, \\ \overline{p}_{i[lk]} + q_{i[l} a_{k]} = 0, \\ \widetilde{q}_{i[lk]} - q_{i[l} a_{k]} = 0, \\ a_m b^m_{jkl} - \widetilde{p}_{jkl} + \overline{q}_{jlk} = 0. \end{array} \right. \tag{10}$$

In addition, upon differentiating conditions (5) and applying equations (9), we find other conditions for the coefficients in (9):

$$\left\{ \begin{array}{ll} \overline{b}^i_{[j|l|k]m} = \delta^i_{[k} \overline{p}_{j]lm}, & \widetilde{b}^i_{[j|l|k]m} = \delta^i_{[k} \widetilde{p}_{j]lm}, \\ \overline{b}^i_{[jk]lm} = \delta^i_{[k} \overline{q}_{j]lm}, & \widetilde{b}^i_{[jk]lm} = \delta^i_{[k} \widetilde{q}_{j]lm}. \end{array} \right. \tag{11}$$

It follows from conditions (5) and (10) that in a fourth-order neighborhood of $x \in M^4$, there are 6 independent components of the tensors $\overline{p}_{ijk}, \widetilde{p}_{ijk}, \overline{q}_{ijk}, \widetilde{q}_{ijk}$ and also 20 independent components of the tensors $\overline{b}^i_{jkl}, \widetilde{b}^i_{jkl}$.

2. For a web $W(3,2,2)$, a transversally geodesic distribution is defined (cf. [AS 92], Section **3.1**) by the equations

$$\xi^2 \underset{1}{\omega}^1 - \xi^1 \underset{1}{\omega}^2 = 0, \;\; \xi^2 \underset{2}{\omega}^1 - \xi^1 \underset{2}{\omega}^2 = 0.$$

If we take $\dfrac{\xi^1}{\xi^2} = -\dfrac{a_2}{a_1}$, we obtain the invariant transversal distribution Δ defined by the equations

$$a_1 \underset{1}{\omega}^1 + a_2 \underset{1}{\omega}^2 = 0, \;\; a_1 \underset{2}{\omega}^1 + a_2 \underset{2}{\omega}^2 = 0. \tag{12}$$

This distribution is defined by the 1-forms

$$\underset{\alpha}{\omega} = a_1 \underset{\alpha}{\omega}^1 + a_2 \underset{\alpha}{\omega}^2, \;\; \alpha = 1, 2. \tag{13}$$

It is connected with a web invariantly and intrinsically since it is defined by the torsion tensor of a web. We will call the distribution Δ the *transversal a-distribution* of a web $W(3,2,2)$. Note that for isoclinicly geodesic webs $W(3,2,2)$, for which $a_1 = a_2 = 0$, the distribution Δ is not defined.

The following theorem gives the conditions of integrability of the distribution Δ.

Theorem 1 *The transversal a-distribution Δ defined by the equations* (12) *is integrable if and only if*

$$\left\{ \begin{array}{l} a_2^2 p_{11} - 2a_1 a_2 p_{(12)} + a_1^2 p_{22} = 0, \\ a_2^2 q_{11} - 2a_1 a_2 q_{(12)} + a_1^2 q_{22} = 0. \end{array} \right. \tag{14}$$

Proof. A transversal distribution Δ defined by equations (12) is integrable if and only if

$$d\underset{\alpha}{\omega} \wedge \underset{1}{\omega} \wedge \underset{2}{\omega} = 0, \qquad \alpha = 1, 2. \tag{15}$$

By (2) and (13), equations (15) take the forms:

$$\left\{ \begin{array}{l} (a_2 \nabla a_1 - a_1 \nabla a_2) \wedge \underset{1}{\omega}^1 \wedge \underset{1}{\omega}^2 \wedge (a_1 \underset{2}{\omega}^1 + a_2 \underset{2}{\omega}^2) = 0, \\ (a_2 \nabla a_1 - a_1 \nabla a_2) \wedge \underset{2}{\omega}^1 \wedge \underset{2}{\omega}^2 \wedge (a_1 \underset{1}{\omega}^1 + a_2 \underset{1}{\omega}^2) = 0, \end{array} \right.$$

where $\nabla a_i = da_i - a_j \omega_i^j$. By (4) and the linear independence of the forms $\underset{\alpha}{\omega}^i$, the last equations imply conditions (14). ■

Note that for an arbitrary web $W(3,2,2)$, it is always possible to take a specialized frame in which there is a relation between the components a_1 and a_2 of the covector a. For example, if the transversal distribution Δ coincides with the distribution $\underset{\alpha}{\omega}^1 = 0$ or $\underset{\alpha}{\omega}^2 = 0$ or $\underset{\alpha}{\omega}^1 + \underset{\alpha}{\omega}^2 = 0$ or $\underset{\alpha}{\omega}^1 - \underset{\alpha}{\omega}^2 = 0$, then we have $a_2 = 0$ or $a_1 = 0$ or $a_1 = a_2$ or $a_1 = -a_2$, respectively. Note that in these cases the forms ω_2^1, ω_1^2, $\omega_1^1 + \omega_1^2 - \omega_2^1 - \omega_2^2$, and $-\omega_1^1 + \omega_1^2 - \omega_2^1 + \omega_2^2$, respectively, are expressed in terms of the basis forms $\underset{\alpha}{\omega}^i$, i.e., in these cases we have

$$\pi_2^1 = 0, \;\; \pi_1^2 = 0, \;\; \pi_1^1 + \pi_1^2 - \pi_2^1 - \pi_2^2 = 0, \;\; -\pi_1^1 + \pi_1^2 - \pi_2^1 + \pi_2^2 = 0,$$

respectively, where $\pi_i^j = \omega_i^j \Big|_{\underset{\alpha}{\omega}^i = 0}$.

In proving the existence theorems it is convenient to use one of these specializations. Let us reformulate Theorem 1 for the first specialization indicated above.

Corollary 2 *If the frame bundle associated with a three-web $W(3,2,2)$ is specialized in such a way that*

$$a_2 = 0, \tag{16}$$

then the a-distribution Δ coincides with the coordinate distribution $\underset{\alpha}{\omega}^1 = 0$ and the condition $\pi_2^1 = 0$ holds. In such a frame bundle the a-distribution is integrable if and only if

$$p_{22} = 0, \;\; q_{22} = 0. \tag{17}$$

Proof. This follows from equations (14) and (16). ■

Each of the relations (14) and (17) gives two conditions which Pfaffian derivatives p_{ij} and q_{ij} of the covector a must satisfy in order for the transversal distribution Δ of a web $W(3,2,2)$ to be integrable.

3. We will now prove an existence theorem for webs with integrable transversal a-distributions Δ.

Theorem 3 *The webs with integrable transversal a-distributions Δ exist, and a solution of a system of differential equations defining such webs depends on five arbitrary functions of three variables.*

Proof. Suppose that specialization (16) has been made. Since our web $W(3,2,2)$ has the integrable a-distribution Δ defined by the equations $\underset{\alpha}{\omega}^1 = 0$, we have conditions (16) and (17), and equations (4) take the form

$$\begin{cases} da_1 - a_1\omega_1^1 = p_{1j}\underset{1}{\omega}^j + q_{1j}\underset{2}{\omega}^j, \\ -a_1\omega_2^1 = p_{21}\underset{1}{\omega}^1 + q_{21}\underset{2}{\omega}^1. \end{cases} \tag{18}$$

The exterior cubic and quadratic equations (7) and (8) become

$$[\nabla b^i_{jkl} + b^i_{jkl} a_1(\underset{1}{\omega}^1 - \underset{2}{\omega}^1] \wedge \underset{1}{\omega}^k \wedge \underset{2}{\omega}^l = 0, \tag{19}$$

$$\begin{cases} (\nabla p_{1k} + p_{1k}a_1\underset{1}{\omega}^1) \wedge \underset{1}{\omega}^k + (\nabla q_{1k} - q_{1k}a_1\underset{2}{\omega}^1) \wedge \underset{2}{\omega}^k + a_1 b^1_{1kl}\underset{1}{\omega}^k \wedge \underset{2}{\omega}^l = 0, \\ \nabla p_{21} \wedge \underset{1}{\omega}^1 + \nabla q_{21} \wedge \underset{2}{\omega}^1 + a_1 b^1_{2kl}\underset{1}{\omega}^k \wedge \underset{2}{\omega}^l = 0. \end{cases} \tag{20}$$

First note that the last equation of (20) implies that

$$b^1_{222} = 0. \tag{21}$$

By (17), (21), and (5), the number of unknown 1-forms (6 forms $\nabla p_{1i}, \nabla q_{1i}$, $\nabla p_{21}, \nabla q_{21}$ and 7 forms among the forms ∇b^i_{jkl}) is 13, $q = 13$ (see [BCGGG 91]).

Since we have 2 exterior quadratic equations and 4 exterior cubic equations (see (19) and (20)), the Cartan's characters are: $s_1 = 2, s_2 = 6$, and $s_3 = 13 - 8 = 5$. As a result, we have $Q = s_1 + 2s_2 + 3s_3 = 29$.

By (10), 13 Pfaffian derivatives of the functions p_{1i}, q_{1i}, p_{21} and q_{21} are independent: 3 functions $\overline{p}_{1jk}$, 4 functions $\widetilde{p}_{1jk}$, 3 functions $\widetilde{q}_{1jk}$, and 3 functions $\overline{p}_{211}, \widetilde{p}_{211}, \widetilde{q}_{211}$. In addition, by (5), (10), and (11), there are 16 independent functions $\overline{b}^i_{jklm}$ and $\widetilde{b}^i_{jklm}$: 10 functions among $\overline{b}^2_{jklm}, \widetilde{b}^2_{jklm}$ and 6 functions $\overline{b}^1_{1111}, \overline{b}^1_{1112}, \overline{b}^1_{1122}, \widetilde{b}^1_{1111}, \widetilde{b}^1_{1112}, \widetilde{b}^1_{1122}$. This implies that the general third-order integral element depends on $N = 13 + 16 = 29$ parameters.

Thus, we have $Q = N$. As a result, the system defining three-webs $W(3,2,2)$ is in involution, and its solution depends on five arbitrary functions of three variables (see [BCGGG 91]). ■

2 Geodesicity of integral surfaces

1. Suppose that the specialization of frames indicated in Section **1** has been made, i.e., we have condition (16): $a_2 = 0$. Then by (12), the distribution Δ is determined by the system of equations

$$\underset{1}{\omega}^1 = 0, \;\; \underset{2}{\omega}^1 = 0. \tag{22}$$

In $T_x(M)$, consider a vectorial frame $\{e^\alpha_i\}$ that is conjugate to the coframe $\{\underset{\alpha}{\omega}^i\}$. Thus for $x \in M$, we obtain

$$dx = e^\alpha_i \underset{\alpha}{\omega}^i.$$

Then on integral surfaces V^2 of the a-distribution Δ, we have

$$dx = e^1_2 \underset{1}{\omega}^2 + e^2_2 \underset{2}{\omega}^2. \tag{23}$$

The 1-forms $\underset{1}{\omega}^2$ and $\underset{2}{\omega}^2$ are basis forms on surfaces V^2, and the vectors e^1_2 and e^2_2 are tangent to these surfaces.

Consider the affine connections Γ defined on the manifold M^4 by 1-forms

$$\theta^u_v = \begin{pmatrix} \theta^i_j & 0 \\ 0 & \theta^i_j \end{pmatrix}, \quad i,j = 1,2; \;\; u,v = 1,2,3,4, \tag{24}$$

where

$$\theta^i_j = \omega^i_j + a^i_{jk}(p\underset{1}{\omega}^k + q\underset{2}{\omega}^k) \tag{25}$$

(see [AS 92], p. 35). For the three-web $W(3,2,2)$ in question, by (6) and (16), formulas (25) take the form:

$$\left\{ \begin{array}{ll} \theta^1_1 = \omega^1_1, & \theta^2_1 = \omega^2_1 + \frac{1}{2}a_1(p\underset{1}{\omega}^2 + q\underset{2}{\omega}^2), \\ \theta^1_2 = \omega^1_2, & \theta^2_2 = \omega^2_2 + \frac{1}{2}a_1(p\underset{1}{\omega}^1 + q\underset{2}{\omega}^1). \end{array} \right. \tag{26}$$

When the vectorial frame $\{e_i^\alpha\}$ moves along the manifold M^4 endowed with a connection Γ, we obtain

$$\left\{ \begin{array}{l} de_2^1 = \theta_2^1 e_1^1 + \theta_2^2 e_2^1, \\ de_2^2 = \theta_2^1 e_1^2 + \theta_2^2 e_2^2. \end{array} \right. \tag{27}$$

We will now prove the following result.

Theorem 4 **(i)** *If the a-distribution Δ is integrable on the web $W(3,2,2)$, then its integral surfaces V^2 are totally geodesic on M^4 in any affine connection of the bundle* (24)–(25).

(ii) *If on a web $W(3,2,2)$ the condition*

$$\omega_2^1 = 0 \tag{28}$$

holds, then the integral surfaces V^2 of the a-distribution Δ are geodesicly parallel in any affine connection of the bundle (24)–(25).

Proof.

(i) By the second of relations (18) and (22), on surfaces V^2 we have

$$\omega_2^1|_{V^2} = 0. \tag{29}$$

This and equations (22) and (26) imply that on V^2 equations (27) take the form

$$de_2^1 = \theta_2^2 e_2^1, \;\; de_2^2 = \theta_2^2 e_2^2. \tag{30}$$

It follows that the bivectors $\Delta = e_2^1 \wedge e_2^2$ are geodesicly parallel on V^2 in any affine connection of the bundle (24)–(25). As a result, the integral surfaces V^2 are totally geodesic on M^4 in any of these connections.

(ii) If equations (28) hold on the entire manifold M^4, then equations (30) are identically satisfied on M^4. Therefore, the bivectors $\Delta = e_2^1 \wedge e_2^2$ are geodesicly parallel on the entire manifold M^4 in any affine connection of the bundle (24)-(25). As a result, the integral surfaces V^2 of the a-distribution Δ are not only totally geodesic but also geodesicly parallel on the entire manifold M^4 in all these connections.

2. Preliminary considerations show that three-webs $W(3,2,2)$, for which integral surfaces V^2 of the transversal a-distribution Δ are geodesicly parallel, exist, and a solution of a system defining such webs depends on four arbitrary functions of three variables. However, we were not able to check the Cartan test in detail.

3. Conditions (28) for integral surfaces V^2 of the transversal a-distribution Δ to be geodesicly parallel were obtained in a specialized frame, i.e., for $a_2 = 0$. To find these conditions in the general frame, we first note that by (18), equations (28) are equivalent to equations

$$p_{21} = 0, \;\; q_{21} = 0 \tag{31}$$

Of course, conditions (17) of integrability of the a-distribution Δ in a specialized frame must be added to conditions (31).

In order to write equations (17) and (31) in the general frame, we write equations (13) in the form

$$\underset{\alpha}{\omega}^{1'} = a_1 \underset{\alpha}{\omega}^1 + a_2 \underset{\alpha}{\omega}^2, \tag{32}$$

consider a relation

$$\underset{\alpha}{\omega}^{2'} = c_1 \underset{\alpha}{\omega}^1 + c_2 \underset{\alpha}{\omega}^2, \tag{33}$$

along with equation (32), and assume that

$$D = \det \begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} \neq 0. \tag{34}$$

The 1-forms $\underset{\alpha}{\omega}^{1'}$ and $\underset{\alpha}{\omega}^{2'}$ form a basis on a manifold M^4 carrying a three-web $W(3,2,2)$ whose coordinate bivectors determined by the equations $\underset{\alpha}{\omega}^{1'} = 0$ and $\underset{\alpha}{\omega}^{2'} = 0$ are transversal. The first of these bivectors is defined by the torsion tensor of the three-web $W(3,2,2)$, and the second one is chosen arbitrarily.

Let us write equations (32) and (33) in the form

$$\underset{\alpha}{\omega}^{i'} = a_j^{i'} \underset{\alpha}{\omega}^j, \tag{35}$$

where the matrix

$$A = (a_j^{i'}) = \begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} \tag{36}$$

is nondegenerate. Its inverse matrix can be written in the form

$$A^{-1} = (a_{i'}^j) = \frac{1}{D} \begin{pmatrix} c_2 & -a_2 \\ -c_1 & a_1 \end{pmatrix}. \tag{37}$$

Under the coframe transformation (35), the tensors p_{ij} and q_{ij} of the web $W(3,2,2)$ undergo the regular tensor transformation:

$$p_{i'j'} = a_{i'}^i a_{j'}^j p_{ij}, \quad q_{i'j'} = a_{i'}^i a_{j'}^j q_{ij}. \tag{38}$$

Taking into account (37), we write formulas (38) for the components p_{21}, q_{21}, p_{22} and q_{22} of these tensors:

$$\begin{cases} p_{2'1'} = c_1(a_2 p_{12} - a_1 p_{22}) + c_2(a_1 p_{21} - a_2 p_{11}), \\ q_{2'1'} = c_1(a_2 q_{12} - a_1 q_{22}) + c_2(a_1 q_{21} - a_2 q_{11}), \\ p_{2'2'} = -a_1(a_2 p_{12} - a_1 p_{22}) - a_2(a_1 p_{21} - a_2 p_{11}), \\ q_{2'2'} = -a_1(a_2 q_{12} - a_1 q_{22}) - a_2(a_1 q_{21} - a_2 q_{11}). \end{cases} \tag{39}$$

Note that the right-hand sides of the last two expressions differ from the left-hand sides of equations (14) only by sign.

Conditions (39) imply the following result.

Theorem 5 *The integral surfaces V^2 of the a-distribution Δ are geodesicly parallel with respect to any affine connection of the bundle* (24)–(25) *if and only if the components of the covector $a = \{a_i\}$ and of the tensors p_{ij} and q_{ij} satisfy the following conditions:*

$$\left\{\begin{array}{ll} a_2 p_{12} - a_1 p_{22} = 0, & a_1 p_{21} - a_2 p_{11} = 0, \\ a_2 q_{12} - a_1 q_{22} = 0, & a_1 q_{21} - a_2 q_{11} = 0. \end{array}\right. \tag{40}$$

Proof. In fact, by (17) and (31), necessary and sufficient conditions for integral surfaces V^2 of the transversal a-distribution Δ to be geodesicly parallel in the general frame with respect to any affine connection of the bundle (24)–(25) have the form

$$p_{2'1'} = 0, \;\; q_{2'1'} = 0, \;\; p_{2'2'} = 0, \;\; q_{2'2'} = 0. \tag{41}$$

But by conditions (34) and (39), equations (41) are equivalent to conditions (40). ■

3 Hexagonality of two-dimensional three-subwebs

1. On integral surfaces of the a-distribution Δ defined on M^4 by the torsion tensor of a web $W(3,2,2)$, the leaves of this web cut two-dimensional three-subwebs $W(3,2,1)$. Let us find the structure equations of these subwebs.

In a specialized frame in which condition (16) holds, the integral surfaces V^2 are defined by the system of equations (22). In addition, the complete integrability of this system on a surface V^2 and equations (31) imply that equation (28) holds. Thus, on a surface V^2, we have

$$\underset{1}{\omega}^1 = 0, \;\; \underset{2}{\omega}^1 = 0, \;\; \omega_2^1 = 0, \tag{42}$$

and the forms $\underset{1}{\omega}^2$ and $\underset{2}{\omega}^2$ are basis forms on V^2. One-dimensional foliations of a web $W(3,2,1)$ are defined on V^2 by the equations

$$\underset{1}{\omega}^2 = 0, \;\; \underset{2}{\omega}^2 = 0, \;\; \underset{1}{\omega}^2 + \underset{2}{\omega}^2 = 0. \tag{43}$$

To find the structure equations of webs $W(3,2,1)$ on surfaces V^2, we substitute the values (31) of the forms $\underset{1}{\omega}^1$, $\underset{2}{\omega}^1$, and ω_2^1 into equations (2) and (3). As a result, we obtain the following structure equations:

$$\left\{\begin{array}{l} d\underset{1}{\omega}^2 = \underset{1}{\omega}^2 \wedge \omega_2^2, \\ d\underset{2}{\omega}^2 = \underset{2}{\omega}^2 \wedge \omega_2^2, \\ d\omega_2^2 = b_{222}^2 \underset{1}{\omega}^2 \wedge \underset{2}{\omega}^2. \end{array}\right. \tag{44}$$

Comparing these equations with the structure equations of a two-dimensional three-web (see [AS 92], p. 18), we see that the form ω_2^2 is the connection form

of the web $W(3,2,1)$, and the component b^2_{222} of the curvature tensor of the web $W(3,2,2)$ is the curvature of the web $W(3,2,1)$:

$$K = b^2_{222}.$$

Since the vanishing of the curvature of the web $W(3,2,1)$ is a necessary and sufficient condition for its hexagonality, we arrive at the following result.

Theorem 6 *Two-dimensional three-webs $W(3,2,1)$ cut on integral surfaces of the a-distribution Δ by the foliations of $W(3,2,2)$ are hexagonal if and only if in the specialized frame bundle defined by condition (16), the component b^2_{222} of the curvature tensor of the web $W(3,2,2)$ vanishes.*

2. We will now prove an existence theorem for webs, for which two-dimensional three-webs $W(3,2,1)$ cut on integral surfaces V^2 by the foliations of the web $W(3,2,2)$ are hexagonal.

Theorem 7 *The webs $W(3,2,2)$, for which two-dimensional three-webs $W(3,2,1)$ cut on integral surfaces V^2 by the foliations of the web $W(3,2,2)$ are hexagonal, exist, and a solution of a system of differential equations defining such webs depends on four arbitrary functions of three variables.*

Proof. Suppose that specialization (16) has been made. Then $a_2 = 0$. Since the a-distribution Δ is integrable, we have conditions (17). As a result, equations (4) take the form (18). As we showed in the proof of Theorem 3, the second of equations (18) implies (21).

Finally, since two-dimensional three-webs $W(3,2,1)$ cut on integral surfaces V^2 by the foliations of the web $W(3,2,2)$, are hexagonal, we have

$$b^2_{222} = 0. \tag{45}$$

By (17), (21), (45), and (5), there are 4 exterior cubic equations (7) and two exterior quadratic equation (8).

By (17), (21), and (5), the number of unknown 1-forms (6 forms $\nabla p_{1i}, \nabla q_{1i}$, $\nabla p_{21}, \nabla q_{21}$ and 6 forms ∇b^i_{jkl}, namely, the forms $\nabla b^1_{111}, \nabla b^1_{112}, \nabla b^1_{122}, \nabla b^2_{111}$, $\nabla b^2_{112}, \nabla b^2_{122}$) is 12, $q = 18$ (see [BCGGG 91]).

Thus, the Cartan's characters are: $s_1 = 2, s_2 = 6$, and $s_3 = 12 - 8 = 4$. As a result, we have $Q = s_1 + 2s_2 + 3s_3 = 26$.

By (10) and (45), 14 Pfaffian derivatives of the functions p_{1i} and q_{1i} are independent: 4 functions $\overline{p}_{111}, \overline{p}_{112}, \overline{p}_{122}, \overline{p}_{211}$, 4 functions $\widetilde{p}_{1jk}$, and 4 functions $\widetilde{q}_{111}, \widetilde{q}_{112}, \widetilde{q}_{122}, \widetilde{q}_{211}$. In addition, by (5), (10), (11), and (45), there are 12 independent functions among $\overline{b}^i_{jklm}$ and $\widetilde{b}^i_{jklm}$: $\overline{b}^i_{1111}, \overline{b}^i_{1112}, \overline{b}^i_{1122}$, $\widetilde{b}^i_{1111}, \widetilde{b}^i_{1112}, \widetilde{b}^i_{1122}$. This implies that the general third-order integral element depends on $N = 14 + 12 = 26$ parameters.

Thus, we have $Q = N$. As a result, the system defining three-webs, for which and two-dimensional three-webs $W(3,2,1)$ cut on integral surfaces V^2 by the foliations of the web $W(3,2,2)$ are hexagonal, is in involution, and its solution depends on four arbitrary functions of three variables (see [BCGGG 91]). ■

3. We will now prove an existence theorem for webs, for which integral surfaces V^2 of the transversal distribution Δ are geodesicly parallel and two-dimensional three-webs $W(3,2,1)$ cut on V^2 by the foliations of the web $W(3,2,2)$, are hexagonal.

Theorem 8 *The webs $W(3,2,2)$, for which integral surfaces V^2 of the transversal distribution Δ are geodesicly parallel, and two-dimensional three-webs $W(3,2,1)$ cut on V^2 by the foliations of the web $W(3,2,2)$, are hexagonal, exist, and a solution of a system of differential equations defining such webs depends on three arbitrary functions of three variables.*

Proof. Suppose that specialization (16) has been made. Then $a_2 = 0$. Since the surfaces V^2 are geodesicly parallel, we have conditions (28), i.e., we have $\omega_2^1 = 0$. As a result, equations (4) take the form

$$\begin{cases} da_1 - a_1\omega_1^1 = p_{1j}\underset{1}{\omega}^j + q_{1j}\underset{2}{\omega}^j, \\ \omega_2^1 = 0. \end{cases} \tag{46}$$

By (4), the second of equations (46) implies that

$$p_{2i} = 0, \quad q_{2i} = 0, \tag{47}$$

and by (3), the same equation implies that

$$b^1_{2kl} = 0. \tag{48}$$

Since two-dimensional three-webs $W(3,2,1)$ cut on V^2 by the foliations of the web $W(3,2,2)$, are hexagonal, we have condition (45):

$$b^2_{222} = 0.$$

Note that conditions (47) imply conditions (17) of integrability of the distribution Δ defined by the equations $\underset{\alpha}{\omega}^1 = 0$.

By (47), (48), (45), and (5), there are 3 exterior cubic equations (7) and only one exterior quadratic equation (8).

By (47), (48), and (5), the number of unknown 1-forms (4 forms $\nabla p_{1i}, \nabla q_{1i}$ and 4 forms ∇b^i_{jkl}, namely, the forms $\nabla b^1_{111}, \nabla b^2_{111}, \nabla b^2_{112}, \nabla b^2_{122}$) is 8, $q = 8$ (see [BCGGG 91]).

Thus, the Cartan's characters are: $s_1 = 1, s_2 = 4$, and $s_3 = 8 - 5 = 3$. As a result, we have $Q = s_1 + 2s_2 + 3s_3 = 18$.

By (10) and (45), 10 Pfaffian derivatives of the functions p_{1i} and q_{1i} are independent: 3 functions $\overline{p}_{111}, \overline{p}_{112}, \overline{p}_{122}$, 4 functions $\widetilde{p}_{1jk}$, and 3 functions $\widetilde{q}_{111}, \widetilde{q}_{112}$, $\widetilde{q}_{122}$, In addition, by (5), (10), (11), (45), and (48), there are 8 independent functions among $\overline{b}^i_{jklm}$ and $\widetilde{b}^i_{jklm}$: $\overline{b}^1_{1111}, \overline{b}^2_{1111}, \overline{b}^2_{1112}, \overline{b}^2_{1122}, \overline{b}^2_{1222}, \overline{b}^2_{2222}, \widetilde{b}^1_{1111}, \widetilde{b}^2_{1111}$. This implies that the general third-order integral element depends on $N = 10 + 8 = 18$ parameters.

Thus, we have $Q = N$. As a result, the system defining three-webs, for which integral surfaces V^2 of the transversal a-distribution Δ are geodesicly parallel, and two-dimensional three-webs $W(3,2,1)$ cut on V^2 by the foliations of the web $W(3,2,2)$ are hexagonal, is in involution, and its solution depends on three arbitrary functions of three variables (see [BCGGG 91]). ■

4. Theorem 6 does not give a condition for two-dimensional three-webs $W(3,2,1)$ cut on integral surfaces V^2 of the transversal a-distribution Δ by the foliations of the web $W(3,2,2)$ to be hexagonal in the general frame. To find such a condition in the general frame, we note that under the coframe transformation (34), the curvature tensor of the web $W(3,2,2)$ undergoes the regular tensor transformation:

$$b^{i'}_{j'k'l'} = a^{i'}_i a^j_{j'} a^k_{k'} a^l_{l'} b^i_{jkl}. \tag{49}$$

We write formulas (49) for the components $b^{1'}_{2'2'2'}$ and $b^{2'}_{2'2'2'}$ of the curvature tensor:

$$b^{1'}_{2'2'2'} = a^{1'}_i b^i, \;\; b^{2'}_{2'2'2'} = a^{2'}_i b^i, \tag{50}$$

where we denote by b^i the following contraction:

$$b^i = b^i_{jkl} a^j_{2'} a^k_{2'} a^l_{2'}.$$

By (37), this contraction can be written as

$$b^i = \frac{1}{D^3}(-b^i_{111} a^3_2 + 3b^i_{(112)} a^2_2 a_1 - 3b^i_{(122)} a_2 a^2_1 + b^i_{222} a^3_1). \tag{51}$$

Equation (51) shows that the contraction b^i is expressed only in terms of components of the torsion and curvature tensors of the web $W(3,2,2)$; that is, b^i is completely determined by this web.

We will now prove the following result.

Theorem 9 *Let $W(3,2,2)$ be a four-dimensional three-web with a nonvanishing covector a and with the integrable transversal a-distributions Δ. Two-dimensional three-webs $W(3,2,1)$ cut on integral surfaces V^2 of the transversal a-distribution Δ by the foliations of the web $W(3,2,2)$ are hexagonal if and only if the torsion and curvature tensors of this web are connected by the relations*

$$b^1 = 0, \;\; b^2 = 0. \tag{52}$$

Proof. Since the a-distribution Δ is integrable, then in the specialized frame condition (21) holds. The hexagonality of the webs $W(3,2,1)$ implies that in the specialized frame $b^2_{222} = 0$. In the general frame these two conditions have the form

$$b^{1'}_{2'2'2'} = 0, \;\; b^{2'}_{2'2'2'} = 0. \tag{53}$$

By (36), (37), and (50), equations (53) can be written as follows:

$$a_1 b^1 + a_2 b^2 = 0, \;\; c_1 b^1 + c_2 b^2 = 0. \tag{54}$$

Since by (34) $D \neq 0$, equations (54) imply conditions (53). ■

5. A three-web $W(3,2,2)$ defines on a manifold M^4 a conformal structure $CO(2,2)$ whose isotropic cones C_x in the tangent space $T_x(M^4)$ are determined by the equation

$$\underset{1}{\omega}^1 \underset{2}{\omega}^2 - \underset{1}{\omega}^2 \underset{2}{\omega}^1 = 0$$

(see [AG 96], p. 196). Transversal bivectors of the three-web $W(3,2,2)$ form one of two families of planar generators of the cones C_x. These bivectors are defined by equations (12). They can be written in the form

$$\underset{1}{\omega}^1 + t\underset{1}{\omega}^2 = 0, \;\; \underset{2}{\omega}^1 + t\underset{2}{\omega}^2 = 0,$$

where $t = \dfrac{a_2}{a_1}$. On the manifold M^4, these bivectors form a fiber bundle E_α whose base is M^4 and whose one-dimensional fibers are defined by the fiber parameter t.

The relative conformal curvature of these bivectors is defined by the formula

$$C(t) = s^2_{111} t^4 - (3s^2_{112} - s^1_{111}) t^3 + 3(s^2_{122} - 3s^1_{112}) t^2 - (3s^2_{222} - 3s^1_{122}) t - s^1_{222}, \tag{55}$$

where $s^i_{jkl} = b^i_{(jkl)}$ is the symmetrized curvature tensor of the web in question ([AG 96], Ch. 5; see also [K 83, 84, 96]). The vanishing of the quantity $C(t)$ singles out four transversal bivectors on the cone C_x. These bivectors are called *principal*.

Next, on a web $W(3,2,2)$ we consider the following contraction:

$$b = b^i a_i. \tag{56}$$

This quantity is an absolute invariant of a web $W(3,2,2)$. Substituting the values (51) of the quantities b^i into equations (56), we find that

$$\begin{aligned} b = -\tfrac{1}{D^3}\big[& b^2_{111} a_2^4 - (3b^2_{(112)} - b^1_{111}) a_2^3 a_1 + 3(b^2_{(122)} - b^1_{(112)}) a_2^2 a_1^2 \\ & -(b^2_{222} - 3b^1_{122}) a_2 a_1^3 + b^1_{222} a_1^4\big]. \end{aligned} \tag{57}$$

Comparing equations (55) and (57), we easily find that

$$b = -\frac{a_1^4}{D^3} C\Big(\frac{a_2}{a_1}\Big). \tag{58}$$

This means that the invariant b of a web $W(3,2,2)$ differs from the relative conformal curvature of the transversal bivector Δ defined by the torsion tensor of $W(3,2,2)$ only by a factor.

Relations (50) and (56) allow us to prove the following result.

Theorem 10 *Let $W(3,2,2)$ be a four-dimensional three-web with a nonvanishing covector a and with the integrable transversal a-distribution Δ defined by*

this covector. Two-dimensional three-webs $W(3,2,1)$ cut on integral surfaces V^2 of the transversal a-distribution Δ by the foliations of the web $W(3,2,2)$ are hexagonal if and only if the a-distribution Δ is one of four principal transversal distributions of the pseudoconformal structure $CO(2,2)$ associated with the web $W(3,2,2)$.

Proof. *Sufficiency.* Using the same considerations which we used in the proof of Theorem 8, we find that the integrability of Δ and the hexagonality of $W(3,2,1)$ lead to conditions (52). Equations (52) and (56) give $b=0$. By (57), the last condition means that the transversal bivectors of the a-distribution Δ are principal.

Necessity. If the a-distribution Δ is integrable and all bivectors Δ of the pseudoconformal structure $CO(2,2)$ defined by the web $W(3,2,2)$ on M^4 are principal, then we have the first equation of (53), $b^{1'}_{2'2'2'}=0$, and $b=b^i a_i=0$. These two conditions imply that

$$K=b^{2'}_{2'2'2'}=0,$$

i.e., the three-webs $W(3,2,1)$ cut on integral surfaces V^2 of the transversal a-distribution Δ by the foliations of the web $W(3,2,2)$ are hexagonal. ■

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